

时间序列分析 25.1 考试参考答案

2025 年 1 月 12 日

一. 填空题

1. $\frac{\sum_{j=1}^K \cos(w_j(t-s))}{K}$
2. $\{b | -0.5 < b < 1, b \neq 0\}$
3. $\frac{a_0}{1-a_2}; 0; a_2$
4. $\frac{\sigma^2}{2\pi} |1 - 0.6e^{i\lambda}|^2, \lambda \in [-\pi, \pi]; \epsilon_t = \sum_{j=0}^{\infty} 0.6^j X_{t-j} - 2.5\mu$
5. $12; \text{ARIMA}(0, 0, 1) \times (1, 0, 0)_{12}; \sigma^2 \frac{1+\theta^2}{1-\Phi^2}$
6. $-3.635, 4.520$
7. $\sigma^2 \frac{1+b^2+2ab}{1-a^2}$
8. 用 $\{X_t, t \leq n\}$ 对 X_{n+1} 做线性预测误差非 0; 用 $\{X_t, t \leq n\}$ 对 X_{n+k} 做线性预测的误差 σ_k^2 满足 $\lim_{k \rightarrow \infty} \sigma_k^2 = \gamma_0$
9. $\frac{\sigma^2}{2\pi|1-0.9e^{i\lambda}|^2}, \lambda \in [-\pi, \pi]; \frac{\sigma^2(1+2\cos\lambda)^2}{18\pi|1-0.9e^{i\lambda}|^2}, \lambda \in [-\pi, \pi]$

二. 计算题

1

(1)

$$\begin{aligned}
 X_t &= a_1 X_{t-1} + \epsilon_t \\
 &= a_1(a_1 X_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
 &= a_1^2 X_{t-2} + a_1 \epsilon_{t-1} + \epsilon_t \\
 &= \cdots = \sum_{j=0}^{t-1} a_1^j \epsilon_{t-j} + a_1^t X_0 \\
 &= \sum_{j=0}^{t-1} a_1^j \epsilon_{t-j}, \quad t \geq 1
 \end{aligned}$$

(2)

$$\begin{aligned}
 \mu(t) &= E(X_t) = E\left(\sum_{j=0}^{t-1} a_1^j \epsilon_{t-j}\right) = \sum_{j=0}^{t-1} a_1^j E(\epsilon_{t-j}) = 0 \\
 \gamma(t, s) &= Cov\left(\sum_{j=0}^{t-1} a_1^j \epsilon_{t-j}, \sum_{l=0}^{s-1} a_1^l \epsilon_{s-l}\right) \\
 &= \sum_{j=0}^{t-1} \sum_{l=0}^{s-1} a_1^{j+l} Cov(\epsilon_{t-j}, \epsilon_{s-l}) \\
 &= \sigma^2 \sum_{j=0}^{s \wedge t-1} a_1^{2j+|t-s|}
 \end{aligned}$$

(3)

由 (2) 可以看出, $\{X_t, t = 1, 2, \cdots\}$ 的自协方差函数不仅与 $|t - s|$ 有关, 还与 $s \wedge t$ 有关, 所以不平稳。

2

(1)

$\epsilon_t | \epsilon_{t-1}, h_{t-1} = v_t \sqrt{0.02 + 0.03\epsilon_{t-1}^2 + 0.024h_{t-1}}$ 因为 $v_t \sim N(0, 1)$ 且正态分布做线性变换仍然为正态分布, 所以

$$\epsilon_t | \epsilon_{t-1}, h_{t-1} \sim N(0, 0.02 + 0.03\epsilon_{t-1}^2 + 0.024h_{t-1})$$

$$\begin{aligned}
& E(Y_t|X_t = 0.1, \epsilon_{t-1} = 0.6, h_{t-1} = 0.2) \\
&= 0.1 + 0.03 \times 0.1 + E(\epsilon_t|\epsilon_{t-1}, h_{t-1}) \\
&= 0.1 + 0.003 + 0 = 0.103
\end{aligned}$$

(2)

$$\begin{aligned}
& Var(Y_t|X_t = 0.1, \epsilon_{t-1} = 0.6, h_{t-1} = 0.2) \\
&= Var(\epsilon_t|\epsilon_{t-1}, h_{t-1}) \\
&= 0.02 + 0.03 \times 0.6^2 + 0.024 \times 0.2 = 0.0356
\end{aligned}$$

(3)

是正态分布。因为给定 $X_t, h_{t-1}, \epsilon_{t-1}$ 下, ϵ_t 服从正态分布, 所以加上常数 $0.1 + 0.03X_{t-1}$ 是线性变换, 仍然服从正态分布。

3

(1)

$$\begin{aligned}
\gamma(s) &= E[X_{t+s}X_t] \\
&= E[(A \cos(\omega(t+s)) + B \sin(\omega(t+s)))(A \cos \omega t + B \sin \omega t)] \\
&= \sigma^2 \cos \omega s
\end{aligned}$$

(2)

$$\begin{aligned}
a &= \frac{1}{\gamma_0} \gamma_1 = \cos \omega \\
L(X_{n+1}|X_n) &= \cos \omega X_n \\
\sigma_1^2 &= E[(X_{n+1} - \cos \omega X_n)X_{n+1}] \\
&= \gamma_0 - \cos \omega \gamma_1 \\
&= \sigma^2 \sin^2(\omega)
\end{aligned}$$

(3)

$$\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} 1 & \cos w \\ \cos w & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos w \\ \cos 2w \end{pmatrix} \\
&= \frac{1}{\sin^2 w} \begin{pmatrix} 1 & -\cos w \\ \cos w & 1 \end{pmatrix} \begin{pmatrix} \cos w \\ \cos 2w \end{pmatrix} = \begin{pmatrix} 2 \cos w \\ -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
L(X_{n+1}|X_n, X_{n-1}) &= 2 \cos \omega X_n - X_{n-1} \\
\sigma_2^2 &= E[(X_{n+1} - 2 \cos \omega X_n + X_{n-1})X_{n+1}] \\
&= \gamma_0 - 2 \cos \omega \gamma_1 + \gamma_2 \\
&= 0
\end{aligned}$$

(4)

因为 $\sigma_2^2 = 0$ 所以 $L(X_{n+1}|X_n, X_{n-1} \dots X_{n-p+1}) = L(X_{n+1}|X_n, X_{n-1})$, $\sigma_p^2 = \sigma_2^2 = 0$; $p \geq 2$.

我们也可以直接证明。注意到当 $n \geq 3$ 时

$$\begin{aligned}
\sigma^{-2n} \det(\Gamma_n) &= \det \begin{pmatrix} \cos(\omega - \omega) & \cos(\omega - 2\omega) & \cdots & \cos(\omega - n\omega) \\ \cos(2\omega - \omega) & \cos(2\omega - 2\omega) & \cdots & \cos(2\omega - n\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(n\omega - \omega) & \cos(n\omega - 2\omega) & \cdots & \cos(n\omega - n\omega) \end{pmatrix} \\
&= \det \begin{pmatrix} \cos(\omega) \cos(\omega) & \cos(\omega - 2\omega) & \cdots & \cos(\omega - n\omega) \\ \cos(2\omega) \cos(\omega) & \cos(2\omega - 2\omega) & \cdots & \cos(2\omega - n\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(n\omega) \cos(\omega) & \cos(n\omega - 2\omega) & \cdots & \cos(n\omega - n\omega) \end{pmatrix} \\
&\quad + \det \begin{pmatrix} \sin(\omega) \sin(\omega) & \cos(\omega - 2\omega) & \cdots & \cos(\omega - n\omega) \\ \sin(2\omega) \sin(\omega) & \cos(2\omega - 2\omega) & \cdots & \cos(2\omega - n\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(n\omega) \sin(\omega) & \cos(n\omega - 2\omega) & \cdots & \cos(n\omega - n\omega) \end{pmatrix} \\
&= \det(\cos(\omega)) \det \begin{pmatrix} \cos(\omega) & \cos(\omega - 2\omega) & \cdots & \cos(\omega - n\omega) \\ \cos(2\omega) & \cos(2\omega - 2\omega) & \cdots & \cos(2\omega - n\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(n\omega) & \cos(n\omega - 2\omega) & \cdots & \cos(n\omega - n\omega) \end{pmatrix} \\
&\quad + \det(\sin(\omega)) \det \begin{pmatrix} \sin(\omega) & \cos(\omega - 2\omega) & \cdots & \cos(\omega - n\omega) \\ \sin(2\omega) & \cos(2\omega - 2\omega) & \cdots & \cos(2\omega - n\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(n\omega) & \cos(n\omega - 2\omega) & \cdots & \cos(n\omega - n\omega) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \det(\cos(\omega)) \begin{pmatrix} \cos(\omega) & -\sin(\omega) \sin(2\omega) & \cdots & -\sin(\omega) \sin(n\omega) \\ \cos(2\omega) & -\sin(2\omega) \sin(2\omega) & \cdots & -\sin(2\omega) \sin(n\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(n\omega) & -\sin(n\omega) \sin(2\omega) & \cdots & -\sin(n\omega) \sin(n\omega) \end{pmatrix} \\
&+ \det(\sin(\omega)) \begin{pmatrix} \sin(\omega) & \cos(\omega) \cos(2\omega) & \cdots & \cos(\omega) \cos(n\omega) \\ \sin(2\omega) & \cos(2\omega) \cos(2\omega) & \cdots & \cos(2\omega) \cos(n\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(n\omega) & \cos(n\omega) \cos(2\omega) & \cdots & \cos(n\omega) \cos(n\omega) \end{pmatrix}
\end{aligned}$$

注意到上面两个矩阵均列线性相关，故行列式都为零。

所以 $\det(\Gamma_n) = 0, n \geq 3$. 故 $(X_n, X_{n-1}, \dots, X_{n-p+1})$ 线性相关, $L(X_{n+1}|X_n, X_{n-1}, \dots, X_{n-p+1}) = L(X_{n+1}|X_n, X_{n-1}, \dots, X_{n-p+2}) = \dots = L(X_{n+1}|X_n, X_{n-1})$.

4

(1)

$$\hat{X}_t(l) = 0.8\hat{X}_t(l-1) - 0.16\hat{X}_t(l-2)$$

$$\text{故有 } \hat{X}_t(l) - 0.4\hat{X}_t(l-1) = 0.4[\hat{X}_t(l-1) - 0.4\hat{X}_t(l-2)].$$

$$\text{计算可得 } \hat{X}_t(l) = 0.4^l[(l+1)X_t - 0.4lX_{t-1}]$$

(2)

$$\begin{aligned}
E[e_t(l)^2] &= E[e_t(l)(X_{t+l})] \\
&= \gamma_0 - 0.4^l(l+1)\gamma_l + 0.4^{l+1}l\gamma_{l+1}.
\end{aligned}$$

也可以直接展开:

$$\begin{aligned}
E[e_t(l)^2] &= E(X_{t+l} - \hat{X}_t(l))^2 \\
&= E(X_{t+l} - (l+1)0.4^l X_t + l0.4^{l+1} X_{t-1})^2 \\
&= (1 + (l+1)^2 0.4^{2l} + l^2 0.4^{2l+2})\gamma_0 + 2l0.4^{l+1}\gamma_{l+1} \\
&\quad - 2(l+1)0.4^l\gamma_l - 2l(l+1)0.4^{2l+1}\gamma_1
\end{aligned}$$

两式值实际上是相同的。